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# Quantum theory for coupled harmonic oscillators with exponentially changing masses

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**Abstract.** The classical solution is obtained for the system of two coupled harmonic oscillators with exponentially decaying mass. Using the Feynman path-integral method of quantization an exact propagator for the corresponding quantum system is derived.

## 1. Introduction

The interest in exactly solvable problems in quantum physics has increased sharply in the last few years. This is concerned, of course, with the fact that the description of the behaviour of nonconservative physical systems is usually very complicated, but sometimes such systems can be modelled by means of quite a simple Hamiltonian, which leads to standard problems of quantum mechanics. Although Feynman's elegant path-integral formulation [1] offers a general approach for treating quantum-mechanical systems, unfortunately only a few problems can be solved exactly. Exactly integrable systems have always been at the centre of attention in physics and mathematics. The systems of oscillator type have been investigated extensively for more than 60 years. Some of the most interesting early results in the study of time-dependent oscillators are reviewed in the monograph [2]. More recently, very interesting systems have been investigated, such as the driven harmonic oscillator with a quadratic Hamiltonian [3], the time-dependent damped driven harmonic oscillator [4], the harmonic oscillator with exponentially decaying mass [5] and with a strongly pulsating mass [6]. But there are many situations [7] (especially in quantum and classical collective processes), where it is very important to know the exact solutions for a system of coupled oscillators. One such problem was solved exactly by Kyu-Hwang Yeon *et al* [8] for two driven coupled harmonic oscillators with constant frequencies and masses. But unfortunately, such a simple system can provide only too rough a model for the processes with changing effective masses. In this case we need a quite simple but adequate model, such as exponentially changing masses. In the present work we firstly find the solution of the classical equations of motion by transforming the time scale for a system of two coupled oscillators with exponentially decaying masses (section 2) and secondly, carry out the quantization of such a system within the framework of the Feynman approach. The exact propagator is obtained in sections 3 and 4.

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## 2. Classical solution

In this section we consider classically a system of two harmonic oscillators with exponentially decaying masses which are coupled together by means of another spring. We assume that the masses of these oscillators and the three spring constants are all the same and take, for convenience, $m = 1$. The Hamiltonian for such a system takes the form

$$\mathcal{H} = \tfrac{1}{2}(p_1^2 + p_2^2)\,\mathrm{e}^{\gamma t} + \omega^2(q_1^2 + q_2^2 - q_1 q_2). \tag{2.1}$$

The ordinary Hamilton equations give

$$\dot{q}_1 = p_1 \mathrm{e}^{\gamma t} \qquad \dot{q}_2 = p_2 \mathrm{e}^{\gamma t} \tag{2.2}$$

$$\dot{p}_1 = \omega^2(q_2 - 2q_1) \qquad \dot{p}_2 = \omega^2(q_1 - 2q_2). \tag{2.3}$$

Going to the new variables $(q, \dot{q})$ in the usual way, we obtain from equation (2.1) the Lagrangian of our system

$$\mathcal{L} = p_1\dot{q}_1 + p_2\dot{q}_2 - \mathcal{H} = \tfrac{1}{2}(\dot{q}_1^2 + \dot{q}_2^2)\,\mathrm{e}^{-\gamma t} - \omega^2(q_1^2 + q_2^2 - q_1 q_2) \tag{2.4}$$

and the corresponding equations of motion

$$\ddot{q}_1 - \gamma\dot{q}_1 + \omega^2(2q_1 - q_2)\,\mathrm{e}^{\gamma t} = 0 \tag{2.5}$$

$$\ddot{q}_2 - \gamma\dot{q}_2 + \omega^2(2q_2 - q_1)\,\mathrm{e}^{\gamma t} = 0. \tag{2.6}$$

To solve these classical equations of motion, we transform the time scale into $\eta = \exp\gamma t$. Then the system of equations (2.5) and (2.6) becomes

$$\frac{\mathrm{d}^2 q_1}{\mathrm{d}\eta^2} + \frac{\omega^2}{\eta\gamma^2}[2q_1(\eta) - q_2(\eta)] = 0 \tag{2.7}$$

$$\frac{\mathrm{d}^2 q_2}{\mathrm{d}\eta^2} + \frac{\omega^2}{\eta\gamma^2}[2q_2(\eta) - q_1(\eta)] = 0. \tag{2.8}$$

To transform these equations into an integrable form we further introduce the normal coordinates,

$$Q_1 = \frac{1}{\sqrt{2}}(q_1 + q_2) \qquad Q_2 = \frac{1}{\sqrt{2}}(q_2 - q_1). \tag{2.9}$$

Then the system (2.7) and (2.8) is reduced to the following:

$$\frac{\mathrm{d}^2 Q_j}{\mathrm{d}\eta^2} + \frac{\Omega_j^2}{\eta\gamma^2}Q_j = 0 \qquad (j = 1, 2) \tag{2.10}$$

where $\Omega_1 = \sqrt{3}\omega$, $\Omega_2 = \omega$.

Using straightforward substitutions, equations (2.10) can be transformed into system of ordinary Bessel equations. It is easy to show that their solutions are

$$Q_j = \eta^{1/2}\left\{Y_{1j}J_1\left(\frac{2\Omega_j}{\gamma}\eta^{1/2}\right) + Y_{2j}N_1\left(\frac{2\Omega_j}{\gamma}\eta^{1/2}\right)\right\} \tag{2.11}$$

where $Y_{kj}$ are constants, $J_1(x)$ and $N_1(x)$ are Bessel and Neumann functions of the first type, respectively.

Returning to the old coordinates, we obtain the general classical solution of our problem in the following simple form:

$$q_1(t) = \mathrm{e}^{\gamma t/2}\left\{C_1 J_1(\xi) + C_2 N_1(\xi) + C_3 J_1(\sqrt{3}\xi) + C_4 N_1(\sqrt{3}\xi)\right\} \tag{2.12}$$

$$q_2(t) = \mathrm{e}^{\gamma t/2}\left\{C_1 J_1(\xi) + C_2 N_1(\xi) - C_3 J_1(\sqrt{3}\xi) - C_4 N_1(\sqrt{3}\xi)\right\} \tag{2.13}$$

where $C_i$ are constants and

$$\xi = \frac{2\omega}{\gamma}\mathrm{e}^{\gamma t/2}. \tag{2.14}$$

Further, we will quantize our system using the explicit form of the general classical solution (2.12) and (2.13).

## 3. Path-integral approach

In this section we will try to derive a propagator for our system of interest following Feynman's interpretation of quantum mechanics [1]. In the path-integral formulation, the solution of the Schrödinger equation is given as the path-dependent integral equation with propagator $\mathcal{P}$,

$$\psi(q_1, q_2, t) = \int \mathrm{d}q_1'\,\mathrm{d}q_2'\,\mathcal{P}(q_1, q_2; t|q_1', q_2'; t')\,\psi(q_1', q_2', t'). \tag{3.1}$$

Propagator $\mathcal{P}$ is the probability to register our system at the moment $t$ in the state with coordinates $q_1$, $q_2$ if at time $t'$ one was in the state with coordinates $q_1'$, $q_2'$. In the case of a bound system, the propagator is expressed in terms of wavefunctions as

$$\mathcal{P}(q_1, q_2; t|q_1', q_2'; t') = \sum_{n=0}^{\infty} \psi_n(q_1, q_2, t)\,\psi_n^*(q_1', q_2', t'). \tag{3.2}$$

In Feynman's approach we have

$$\mathcal{P}(q_1, q_2; t|q_1', q_2'; t') = \int_{(q_1', q_2', t')}^{(q_1, q_2, t)} \mathcal{D}q(t)\exp\left\{\frac{\mathrm{i}}{\hbar}S(q_1, q_2, q_1', q_2'; t, t')\right\} \tag{3.3}$$

where the measure is given by

$$\mathcal{D}q(t) = \exp\{-\tfrac{1}{2}\gamma(t - t')\}\lim_{N\to\infty}\frac{1}{A}\prod_{j=1}^{N-1}\frac{\mathrm{d}q_{1j}\,\mathrm{d}q_{2j}}{A^2} \tag{3.4}$$

and $S(q_1, q_2, q_1', q_2'; t, t')$ is the action defined as the time integral over the Lagrangian $\mathcal{L}(q_1, q_2, \tau)$ between the time points $t'$ and $t$, i.e.

$$S(q_1, q_2, q_1', q_2'; t, t') = \int_{t'}^{t} \mathrm{d}\tau\,\mathcal{L}(q_1, q_2, \tau). \tag{3.5}$$

Equation (3.4) contains the normalization factor $A$ given by

$$A = (2\pi\mathrm{i}\hbar\epsilon)^{1/2} \qquad \epsilon = \lim_{n\to 0}\frac{t}{N}. \tag{3.6}$$

In equation (3.4) we have accounted for the time dependence of mass (see [9]) and the identity

$$\prod_{j=1}^{N-1}\mathrm{e}^{-\gamma\epsilon} = \mathrm{e}^{-\gamma(t-t')}.$$

After substitution of equation (2.4) into equation (3.5), the action becomes

$$S(q_1, q_2, q_1', q_2'; t, t') = S^{(cl)}(q_1, q_2, q_1', q_2'; t, t') + \int_{t'}^{t}\mathrm{d}\tau\left\{\tfrac{1}{2}\left[\dot{y}_1^2(\tau) + \dot{y}_2^2(\tau)\right]\mathrm{e}^{-\gamma t} - \omega^2\left[y_1^2(\tau) + y_2^2(\tau) - y_1(\tau)y_2(\tau)\right]\right\}. \tag{3.7}$$

Here $S^{(cl)}$ is the classical action and $y_i$ are quantum fluctuations of trajectories $q_i(t)$:

$$y_i = q_i - q_i^{(cl)} \qquad (i = 1, 2) \tag{3.8}$$

where $q_i^{(cl)}$ is a classical path.

Using equation (3.7) the propagator (3.3) can be expressed in the following convenient form:

$$\mathcal{P}(q_1, q_2; t|q_1', q_2'; t') = F(t, t')\, \mathrm{e}^{(\mathrm{i}/\hbar)S^{(cl)}} \tag{3.9}$$

where $F(t, t')$ is the multiplicative function that can be presented as

$$F(t, t') = \int_{(0)}^{(0)} \mathcal{D}y(t) \exp\left\{\frac{\mathrm{i}}{2\hbar}\int_{t'}^{t} \mathrm{d}t \left[(\dot{y}_1^2 + \dot{y}_2^2)\,\mathrm{e}^{-\gamma t} - 2\omega^2(y_1^2 + y_2^2 - y_1 y_2)\right]\right\}. \tag{3.10}$$

Thus, we obtained the result that, as usual, the propagator depends only on the classical action (the multiplicative function does not depend on trajectories). Now we are going to obtain the multiplicative function (3.10) explicitly. For this purpose equation (3.10) should be rewritten in terms of normal coordinates (2.9). Of course, the condition $(y_1, y_2) = (0, 0)$ is reduced automatically to $(z_1, z_2) = (0, 0)$, where

$$z_i = Q_i - Q_i^{(cl)} \qquad (i = 1, 2). \tag{3.11}$$

After such a substitution we come to the equation

$$F(t, t') = J\int_{(0)}^{(0)} \mathcal{D}z(t) \exp\left\{\frac{\mathrm{i}}{2\hbar}\int_{t'}^{t} \mathrm{d}\tau \left[(\dot{z}_1^2\mathrm{e}^{-\gamma t} - \omega^2 z_1^2) + (\dot{z}_2^2\mathrm{e}^{-\gamma t} - 3\omega^2 z_2^2)\right]\right\}. \tag{3.12}$$

The path transformation has an obvious form

$$\begin{pmatrix} y_{k1} \\ y_{k2} \end{pmatrix} = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} z_{k1} \\ z_{k2} \end{pmatrix} \tag{3.13}$$

where $k = 1, 2, \ldots, N$. Therefore, Jacobian $J$ in equation (3.12) becomes unity.

Since we have separated the action into functionals with only the same variables in the path integral, then this integral can be represented by the multiplication of path integrals with each variable. So, we have

$$\begin{aligned} F(t, t') = F_1(t, t')\, F_2(t, t') = &\left\{\int_{(0)}^{(0)} \mathcal{D}z_1(t) \exp\left[\frac{\mathrm{i}}{2\hbar}\int_{t'}^{t} \mathrm{d}\tau\, (\dot{z}_1^2\mathrm{e}^{-\gamma t} - \omega^2 z_1^2)\right]\right\} \\ &\times\left\{\int_{(0)}^{(0)} \mathcal{D}z_2(t) \exp\left[\frac{\mathrm{i}}{2\hbar}\int_{t'}^{t} \mathrm{d}\tau\, (\dot{z}_2^2\mathrm{e}^{-\gamma t} - 3\omega^2 z_2^2)\right]\right\}. \end{aligned} \tag{3.14}$$

Since the integrals in equation (3.14) have a Gaussian form, they can be evaluated using the well developed general methods [10], but here we exploit the simplest one. In view of the form of equation (3.14) (i.e. the product of multiplicative functions for two independent oscillators), one can use the formula (see, e.g., [11])

$$F(t, t') = \frac{1}{2\pi\mathrm{i}\hbar}\left(\frac{\partial^2 S^{(cl)}}{\partial Q_1\, \partial Q_1'}\frac{\partial^2 S^{(cl)}}{\partial Q_2\, \partial Q_2'}\right)^{1/2} \tag{3.15}$$

where $Q_j$ are defined by equations (2.9) and $S^{(cl)}$ is the classical action, which will be evaluated in the next section.

## 4. Classical action and propagator

The classical action for two coupling harmonic oscillators with exponentially decaying mass according to equations (2.4) and (3.5) takes the following form:

$$S^{(cl)} = \int_{t'}^{t} \left[\tfrac{1}{2}\left(\dot{q}_1^{(cl)2} + \dot{q}_2^{(cl)2}\right) \mathrm{e}^{-\gamma\tau} - \omega^2\left(q_1^{(cl)2} + q_2^{(cl)2} - q_1^{(cl)} q_2^{(cl)}\right)\right] \tag{4.1}$$

where $q_i^{(cl)}$ and $\dot{q}_i^{(cl)}$ are the classical path and velocity, respectively (in the following we will drop the superscript $(cl)$). Integrating equation (4.1) over time, we obtain

$$\begin{aligned} S^{(cl)} &= \tfrac{1}{2}(q_1\dot{q}_1 + q_2\dot{q}_2)\,\mathrm{e}^{-\gamma\tau}|_{t'}^{t} - \int_{t'}^{t} \mathrm{d}\tau\, \mathrm{e}^{-\gamma\tau} \frac{q_1}{2}\left[\ddot{q}_1 - \gamma\dot{q}_1 + \omega^2(2q_1 - q_2)\,\mathrm{e}^{\gamma\tau}\right] \\ &\quad - \int_{t'}^{t} \mathrm{d}\tau\, \mathrm{e}^{-\gamma\tau} \frac{q_2}{2}\left[\ddot{q}_2 - \gamma\dot{q}_2 + \omega^2(2q_2 - q_1)\,\mathrm{e}^{\gamma\tau}\right] \\ &= \tfrac{1}{2}\left\{\left[q_1(t)\,\dot{q}_1(t) + q_2(t)\,\dot{q}_2(t)\right]\mathrm{e}^{-\gamma t} - \left[q_1(t')\,\dot{q}_1(t') + q_2(t')\,\dot{q}_2(t')\right]\mathrm{e}^{-\gamma t'}\right\}. \end{aligned} \tag{4.2}$$

Here we have taken into account the classical equations of motion (2.5) and (2.6).

To obtain an exact expression for equation (4.2) we note that classical trajectories $q_i$ are given by the exact solutions (2.12) and (2.13). Using the ordinary differential relations for Bessel and Neumann functions [10], we obtain

$$\dot{q}_1 = \omega \mathrm{e}^{\gamma t/2}\left[C_1 J_0(\xi) + C_2 N_0(\xi) + \sqrt{3}C_3\, J_0(\sqrt{3}\xi) + \sqrt{3}C_4\, N_0(\sqrt{3}\xi)\right] \tag{4.3}$$

$$\dot{q}_2 = \omega \mathrm{e}^{\gamma t/2}\left[C_1 J_0(\xi) + C_2 N_0(\xi) - \sqrt{3}C_3\, J_0(\sqrt{3}\xi) - \sqrt{3}C_4\, N_0(\sqrt{3}\xi)\right]. \tag{4.4}$$

Using equations (2.12) and (2.13) for both $q_i(t)$ and $q_i(t')$ we can express constants $C_j$ as

$$C_1 = \frac{N_1(\xi)}{2\Delta(\xi,\xi')}\left\{(q_1' + q_2')\,\mathrm{e}^{-\gamma t'/2} - \frac{N_1(\xi')}{N_1(\xi)}(q_1 + q_2)\,\mathrm{e}^{-\gamma t/2}\right\} \tag{4.5}$$

$$C_2 = \frac{J_1(\xi')}{2\Delta(\xi,\xi')}\left\{(q_1 + q_2)\,\mathrm{e}^{-\gamma t/2} - \frac{J_1(\xi)}{J_1(\xi')}(q_1' + q_2')\,\mathrm{e}^{-\gamma t'/2}\right\}. \tag{4.6}$$

We do not write the constants $C_3$ and $C_4$ because they have the same form as $C_1$ and $C_2$, respectively, but with the substitutions $\xi \to \sqrt{3}\xi$ and $\xi' \to \sqrt{3}\xi'$.

Using equations (4.2)–(4.6), we finally obtain the classical action in the form

$$\begin{aligned} S^{(cl)} = \tfrac{1}{4}\omega\Big\{& A(\xi,\xi')\left[(q_1+q_2)^2 - (q_1'+q_2')^2\right] + \tilde{A}(\xi,\xi')\left[(q_1-q_2)^2 - (q_1'-q_2')^2\right] \\ &+ (q_1+q_2)(q_1'+q_2')\left[D(\xi,\xi')\,\mathrm{e}^{\frac{1}{2}\gamma(t-t')} + G(\xi,\xi')\,\mathrm{e}^{\frac{1}{2}\gamma(t'-t)}\right] \\ &+ (q_1-q_2)(q_1'-q_2')\left[\tilde{D}(\xi,\xi')\,\mathrm{e}^{\frac{1}{2}\gamma(t-t')} + \tilde{G}(\xi,\xi')\,\mathrm{e}^{\frac{1}{2}\gamma(t'-t)}\right]\Big\} \end{aligned} \tag{4.7}$$

where we have introduced the functions

$$D(\xi,\xi') = \frac{1}{\Delta(\xi,\xi')}\left[J_0(\xi)\,N_1(\xi) - J_1(\xi)\,N_0(\xi)\right] \tag{4.8}$$

$$G(\xi,\xi') = \frac{1}{\Delta(\xi,\xi')}\left[J_0(\xi')\,N_1(\xi') - J_1(\xi')\,N_0(\xi')\right] \tag{4.9}$$

$$A(\xi,\xi') = \frac{1}{\Delta(\xi,\xi')}\left[J_1(\xi')\,N_0(\xi) - J_0(\xi)\,N_1(\xi')\right] \tag{4.10}$$

$$\begin{aligned}\tilde{D}(\xi,\xi') &= \sqrt{3}D(\sqrt{3}\xi,\sqrt{3}\xi')\\ \tilde{G}(\xi,\xi') &= \sqrt{3}G(\sqrt{3}\xi,\sqrt{3}\xi')\\ \tilde{A}(\xi,\xi') &= \sqrt{3}A(\sqrt{3}\xi,\sqrt{3}\xi')\end{aligned} \tag{4.11}$$

and

$$\Delta(\xi,\xi') = J_1(\xi')N_1(\xi) - J_1(\xi)N_1(\xi'). \tag{4.12}$$

In the process of evaluating equations (4.5) and (4.6) we have used the well known recurrent formulae for functions of Bessel type [12].

Now we may return to equation (3.15) and calculate the multiplicative function $F(t,t')$, taking into account definition (2.9). The result is

$$\begin{aligned}F(t,t') = \frac{\omega}{4\pi\hbar}\big\{&D(\xi,\xi')\,\tilde{G}(\xi,\xi') + \tilde{D}(\xi,\xi')\,G(\xi,\xi')\\ &+D(\xi,\xi')\tilde{D}(\xi,\xi')\,\mathrm{e}^{\gamma(t-t')} + G(\xi,\xi')\,\tilde{G}(\xi,\xi')\,\mathrm{e}^{\gamma(t'-t)}\big\}.\end{aligned} \tag{4.13}$$

Thus finally, the exact propagator for the system of two coupled oscillators with exponentially decaying mass is given by equation (3.9), where the multiplicative function $F(t,t')$ is defined by equation (4.13) and the classical action is given by (4.7). The propagator obtained gives all information about our system. Further, we can derive the energy expectation values, exact wavefunctions (starting from equation (3.2)) and the uncertainty relations, etc.

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